Analysis and observer design in synchronization via a state feedback control method

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On the basis of a Lure-type system, a systematic procedure of state feedback control was developed to analyze the synchronization of two chaotic systems. To ensure a stable synchronized result, the conditions of stability are investigated. Moreover, with the aid of a deterministic observer, the unmeasured states are reconstructed and the time of synchronization can be arbitrarily designed with a guaranteed stability. Meanwhile, if the system's output is corrupted by the measurement noise, then a stochastic observer (extended Kalman filter) is proposed to reject the noise. [S1063-651X(97)06711-1]

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I. INTRODUCTION

Synchronization between two identical (or nearly identical) chaotic systems recently has received a great deal of interest in light of its potential applications [1,2]. In particular, the use of chaotic synchronization in communication fields has been investigated by many workers; see e.g., [3–7]. Because of the broadband character of the chaotic signal, it is especially availing in secure signal transmission [8–10]. There are various strategies that have been applied to carry out the synchronization. Some main approaches of those strategies had also been verified by experiments such as Pecora and Carroll's method [1], the Ott-Grebogi-York (OGY) [11]–based method [12], and the state feedback control (SFC) method [13–15]. In the method of Pecora and Carroll, a system is divided into a drive subsystem (whose largest Lyapunov exponent is positive) and a driven subsystem (whose Lyapunov exponents are all negative). Thus the trajectories from two identical driven subsystems can be synchronized if the same driven system is used. Nevertheless, due to the divided requirements of the system, this method is not always applicable. Subsequently, methods that do not require division were employed. First, a method based on the OGY idea was used, which requires continual monitoring of the system's states and deals with the Poincaré map. Thereafter, the SFC method was applied based on the idea of Pyragas [16,17]. In practical applications, due to self-regulation of state feedback, the method of SFC does not need a real-time computer analysis and is particularly favorable for experimentation.

The stability of synchronization is also an important subject attracting many researchers. In the case of Pecora and Carroll's method, a systematic analysis procedure was found in [18] and then a necessary and sufficient condition based on the asymptotic stability was presented [19]. In applications, an appropriate Lyapunov function is needed for the subsystem and the stability of synchronization needs to be investigated in a case by case study. If a suitable Lyapunov function is not found, this method for synchronization then fails. In the case of the OGY-based method, by reason of a restricted application (applied only to the systems described by two-dimensional maps on a Poincaré surface), the discussion of the stability of the method is limited. In the case of the SFC method, the stability was discussed by Pyragas [17]. It dealt with the conditional Lyapunov exponent calculated from a linearized result of the variational equation. Due to the requirements of previous computer analyses, the SFC method suffers in problems of coupled gain selection (or perturbation weight), the number of measurements (or perturbations), and the initial error between two chaotic systems. Until now, most recent papers applying this method still handled those problems with a trial and error gain selection, ambiguous initial error, and numerous computer simulations [13–15]. In fact, the above items affect each other mutually and need an efficient systematic analysis initially.

Due to the state feedback coupling of the SFC method, theoretically, it is always possible to have a stable synchronization result through a coupled gain and coupled state (measurement) adjustment. Briefly, in the error equation of the SFC method, the eigenvalues of the linear part can be designed (a selective synchronization time), which is an apparent distinction from other methods. In this paper we provide a methodical procedure to analyze the SFC method and condense it with a criterion of synchronization stability. Furthermore, under noisy or only output available circumstances, we built an observer to estimate the unmeasured states and to reject the noise. Thereafter, the stability of synchronization still can be accomplished.

The outline of the remainder of this paper is as follows. In Sec. II we briefly discuss the method of the SFC based on the concept of Lure's system. In essence, this point of view is to separate a nonlinear physical system into a linear dynamical system and a nonlinear element. Therefore, the whole system can be represented as a feedback connection that is easily analyzed by a linear theory. An observer design is also introduced in this section to assist the stability of synchronization. In Sec. III two typical examples are used to demonstrate the proposed design procedures. Finally, in Sec. IV we summarize the results of this paper.

II. STATE FEEDBACK CONTROL METHOD

A. Output feedback coupling

Suppose that there are two identical (or nearly identical) chaotic systems in the form of

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x},\mathbf{u}), \quad \dot{\mathbf{y}}=\mathbf{F}(\mathbf{y},\mathbf{u}), \tag{1}$$

where $\mathbf{F}:\mathbf{R}^n\times\mathbf{R}^p\rightarrow\mathbf{R}^n$ is a vector field of the system, $\mathbf{x},\mathbf{y}$

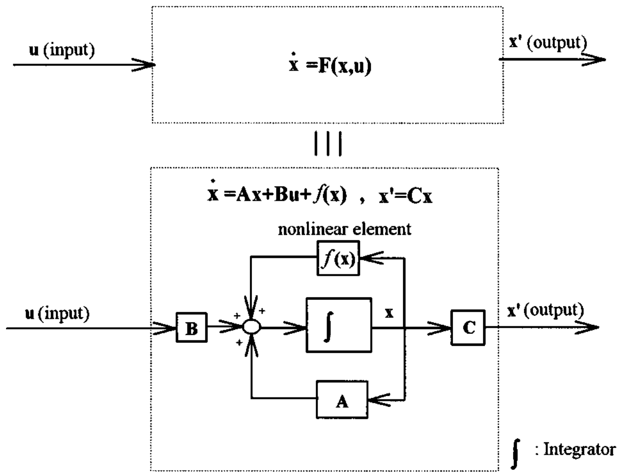

FIG. 1. Nonlinear system represented by a feedback connection of a linear system and a nonlinear element.

$\in \mathbf{R}^n$ are the system states, and $\mathbf{u} \in \mathbf{R}^p$ is the control input. We assumed that the system (1) has a ''*Lure-type system*'' [21] form, i.e., a linear time-invariant (constant) system with a feedback connection of nonlinear element as shown in Fig. 1. On the basis of this point of view, we separated (rearrange) system (1) as into the linear and nonlinear parts

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + f(\mathbf{x}), \quad \mathbf{x}' = \mathbf{C}\mathbf{x}\\ \dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} + f(\mathbf{y}), \quad \mathbf{y}' = \mathbf{C}\mathbf{y},\end{aligned} \tag{2}$$

where the matrix $\mathbf{A} \in \mathbf{R}^{n\times n}$ is the linear parts of the plane (time invariant), $\mathbf{B} \in \mathbf{R}^{n\times p}$ is the control matrix, $\mathbf{x}', \mathbf{y}' \in \mathbf{R}^m$ are the measurements with an output matrix $\mathbf{C} \in \mathbf{R}^{m\times n}$, and $f(\mathbf{x})$ is the nonlinear element. It should be noted that the matrix $\mathbf{A}$ is not a linearized result (i.e., Jacobian linearization) of $\mathbf{F}$ nor is the nonlinear element $f(\mathbf{x})$ a higher-order term of $\mathbf{F}$ in an expansion around a fixed point. A schematic diagram of Eq. (2) is shown in Fig. 2. The process of representing a nonlinear system in Lure form depends on the particular system involved. If the states of the system (1) cannot be separated, we simply set $\mathbf{A}$ to be a zero matrix. Otherwise, if the control term of the system (1) is not separable, then the system fails to have a Lure form. Basically, it is not hard to find a linear time-invariant part of matrix $\mathbf{A}$, whereas there may be some difficulty in separating the term of control force $\mathbf{u}$. Fortunately, in many physical cases including chaotic ones (e.g., the Lorenz system and the Duffing system), it is not difficult to represent the system in the feedback form of Fig. 1. Hence, in Eq. (2) let the $\mathbf{x}$ system be driven (slave) to synchronize the $\mathbf{y}$ system (master). Two chaotic systems are coupled by the difference of the output vector and the matrix of coupled gain as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + f(\mathbf{x}) - \mathbf{B}\mathbf{K}(\mathbf{x}' - \mathbf{y}'), \tag{3a}$$

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} + f(\mathbf{y}), \tag{3b}$$

where $\mathbf{K} \in \mathbf{R}^{p\times m}$ is the matrix of the feedback coupled gain. Note that in Eq. (3a) the coupled term $\mathbf{BK}(\mathbf{y}' - \mathbf{x}')$ without the control vector $\mathbf{B}$ actually acts as a software observer known in engineering fields (a Thau observer [20]). Further, let us define the synchronization error vector by $\tilde{\mathbf{y}} \equiv \mathbf{x} - \mathbf{y}$; then the error dynamics obeys

$$\begin{aligned}\dot{\tilde{\mathbf{y}}} &= \mathbf{A}\tilde{\mathbf{y}} + f(\mathbf{x}) - f(\mathbf{y}) - \mathbf{BKC}(\mathbf{x} - \mathbf{y})\\ &= \mathbf{A}_0\tilde{\mathbf{y}} + f(\mathbf{y} + \tilde{\mathbf{y}}) - f(\mathbf{y}),\end{aligned} \tag{4}$$

where $\mathbf{A}_0 = \mathbf{A} - \mathbf{BKC}$. In Eq. (4) the synchronization of two systems has been condensed to a quasilinear system with the linear part $\mathbf{A}_0$ and nonlinear part $f(\mathbf{y} + \tilde{\mathbf{y}}) - f(\mathbf{y})$. Hence the effects of the initial error between two chaotic systems can be easily understood by the nonlinear part of this equation. Moreover, the effects of the coupled gain $\mathbf{K}$ and number of measurements can be realized by the Hurwitz (i.e., the eigenvalues are in the open left-half plane) notion of the matrix $\mathbf{A}_0$. Subsequently, in order to have a convergent result in Eq. (4), the Lyapunov function is used to guarantee the synchronization stability. First we assume that the nonlinear function $f(\ )$ is confined by a local Lipschitz condition, i.e.,

$$\|f(\mathbf{x}) - f(\mathbf{y})\| = \|f(\mathbf{y} + \tilde{\mathbf{y}}) - f(\mathbf{y})\| \leqslant L\|\tilde{\mathbf{y}}\|, \tag{5}$$

where L is a local Lipschitz constant. In an increasing nonlinear element, the greater the initial error, the larger the Lipschitz constant L (as shown in Sec. III). Actually, this constant can be interpreted as the maximum gradient in the region of interest. Then, if $\mathbf{A}_0$ is Hurwitz, given a positive-definite matrix $\mathbf{Q}$, there is a constant, symmetric, positive-definite matrix $\mathbf{P}$ such that the Lyapunov equation is

$$\mathbf{P}\mathbf{A}_0 + \mathbf{A}_0^T\mathbf{P} = -\mathbf{Q}. \tag{6}$$

Let the quadratic Lyapunov function be $V(\tilde{\mathbf{y}}) = \tilde{\mathbf{y}}^T\mathbf{P}\tilde{\mathbf{y}}$; then the derivative of $V(\tilde{\mathbf{y}})$ along the trajectories of the error system satisfies

$$\begin{aligned}\dot{V}(\tilde{\mathbf{y}}) &= -\tilde{\mathbf{y}}^T\mathbf{Q}\tilde{\mathbf{y}} + 2\tilde{\mathbf{y}}^T\mathbf{P}L\|\tilde{y}\|_2\\ &\leqslant -\lambda_{\min}(\mathbf{Q})\|\tilde{\mathbf{y}}\|_2^2 + 2\lambda_{\max}(\mathbf{P})L\|\tilde{\mathbf{y}}\|_2^2.\end{aligned} \tag{7}$$

The synchronized error converges asymptotically to zero if

$$L < \lambda_{\min}(\mathbf{Q})/2\lambda_{\max}(\mathbf{P}), \tag{8}$$

where λ are the eigenvalues of matrix $\mathbf{Q}$ or $\mathbf{P}$. Without a loss of precesion we set $\mathbf{Q} = \mathbf{I}$ and thus the ratio in Eq. (8) has a maximum value (further details can be found in [21]). Therefore, we have the inequality

$$2\lambda_{\max}(\mathbf{P})L < 1. \tag{9}$$

Since the feedback coupling term $\mathbf{BKC}$ produces a Hurwitz $\mathbf{A}_0$, the rules in the linear system can be applied directly in Eq. (4). If the matrix $\mathbf{C}$ is a fully state measurement and the pair ($\mathbf{A}$,$\mathbf{B}$) is controllable [22] (i.e., through the matrix $\mathbf{B}$ the actuator can excite the system states to any points in the phase space), then the eigenvalues of $\mathbf{A}_0$ can be assigned arbitrarily, i.e., $\lambda_{\max}(\mathbf{P})$ can be chosen. Consequently, the synchronization between the chaotic systems can be accomplished under any initial errors. Here the synchronization problems have been interpreted as a pole placement of the linear part in the quasilinear equation (4). Once the choice of the eigenvalues of $\mathbf{A}_0$ (i.e., the choice of the maximum eigenvalue of $\mathbf{P}$) holds the inequality (9), the synchronization

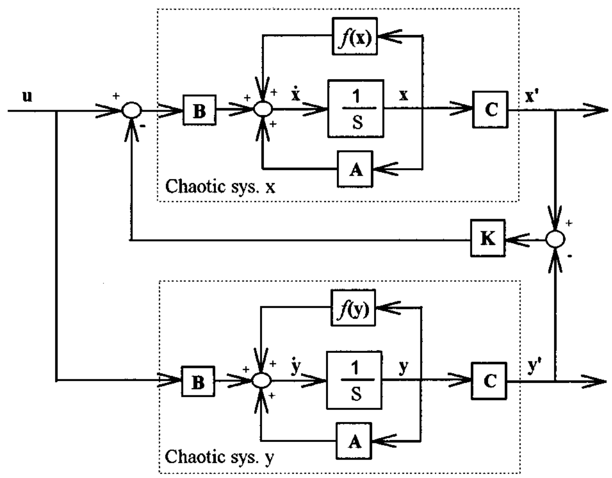

FIG. 2. Synchronization of two chaotic systems interpreted in the ''Lure-type system'' form.

converges asymptotically to zero within the region of local Lipschitz availability. To this end, the influence factors (coupled gain selection, number of measurements, and initial error of two systems) have been systematically connected together and can be summarized as follows.

In order to cover any initial errors, an appropriate maximum eigenvalue of $\mathbf{P}$ is needed (the suitable eigenvalues of $\mathbf{A}_0$), i.e., the matrix $\mathbf{C}$ must be a full state measurement and the pair $(\mathbf{A},\mathbf{B})$ must be controllable. If this is not the case, we are aware that the initial error may not be given arbitrarily and the stability of synchronization is studied with a case-by-case applicability.

B. Observer feedback coupling

1. Deterministic case (without noise)

In the case in which the system's states are not fully accessible, we propose a nonlinear observer (deterministic) to estimate those states. With the same systems and output as in Eq. (2), the observer is designed for each chaotic system to reconstruct the embedded states as shown in Fig. 3. In practicality, the observers are implemented by computer software. If the pair $(\mathbf{A},\mathbf{C})$ is observable [22] (i.e., through the matrix $\mathbf{C}$ the measurement can read out all states in the system), the observer is asymptotically converged. The mathematical formulation is

$$\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{Bu}+f(\mathbf{x})-\mathbf{BK}(\hat{\mathbf{x}}-\hat{\mathbf{y}}), \tag{10a}$$

$$\dot{\hat{\mathbf{x}}}=\mathbf{A}\hat{\mathbf{x}}+\mathbf{Bu}+f(\hat{\mathbf{x}})-\mathbf{BK}(\hat{\mathbf{x}}-\hat{\mathbf{y}})-\mathbf{K}_x(\mathbf{C}\hat{\mathbf{x}}-\mathbf{x}'), \tag{10b}$$

$$\dot{\mathbf{y}}=\mathbf{Ay}+\mathbf{Bu}+f(\mathbf{y}), \tag{11a}$$

$$\dot{\hat{\mathbf{y}}}=\mathbf{A}\hat{\mathbf{y}}+\mathbf{Bu}+f(\hat{\mathbf{y}})-\mathbf{K}_y(\mathbf{C}\hat{\mathbf{y}}-\mathbf{y}'), \tag{11b}$$

where $\hat{\mathbf{x}},\hat{\mathbf{y}}\in\mathbf{R}^n$ are the observer's states and $\mathbf{K}_x,\mathbf{K}_y\in\mathbf{R}^{n\times m}$ are the gain of each observer. If defining the observer's error as $\mathbf{e}_x\equiv\hat{\mathbf{x}}-\mathbf{x}$ and $\mathbf{e}_y\equiv\hat{\mathbf{y}}-\mathbf{y}$, the error dynamics in Eqs. (10) and (11) then obeys

$$\begin{aligned}\dot{\mathbf{e}}_x&=(\mathbf{A}-\mathbf{K}_x\mathbf{C})\mathbf{e}_x+f(\mathbf{x}+\mathbf{e}_x)-f(\mathbf{x}),\\ \dot{\mathbf{e}}_y&=(\mathbf{A}-\mathbf{K}_y\mathbf{C})\mathbf{e}_y+f(\mathbf{y}+\mathbf{e}_y)-f(\mathbf{y}).\end{aligned} \tag{12}$$

Thereafter, the error dynamics of synchronization $(\tilde{\mathbf{y}}\equiv\mathbf{x}-\mathbf{y})$ is

$$\dot{\tilde{\mathbf{y}}}=(\mathbf{A}-\mathbf{BK})\tilde{\mathbf{y}}+f(\mathbf{y}+\tilde{\mathbf{y}})-f(\mathbf{y})-\mathbf{BK}(\mathbf{e}_x-\mathbf{e}_y). \tag{13}$$

With a time-scale decomposition [21], let us first switch on the observer and then the synchronization command. Consequently, the converged time both in the observer and in synchronization can be designed under an appropriate gain selection. Note that the feedback gains in observers $(\mathbf{K}_x,\mathbf{K}_y)$ are software and in synchronization $(\mathbf{K})$ hardware.

2. Stochastic case (with noise)

The observer results in the above deterministic case can be easily extended with a similar procedure to a stochastic case. Taking into account the measurement noise (suppose that there is no input disturbances) in Eqs. (2) and (3), the problems of synchronization between two systems (without observer) are reformulated as

$$\begin{aligned}&\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{Bu}+f(\mathbf{x})-\mathbf{BK}(\mathbf{x}'-\mathbf{y}'),\quad \mathbf{x}'=\mathbf{Cx}+\mathbf{v}_x\\ &\dot{\mathbf{y}}=\mathbf{Ay}+\mathbf{Bu}+f(\mathbf{y}),\quad \mathbf{y}'=\mathbf{Cy}+\mathbf{v}_y,\end{aligned} \tag{14}$$

where $\mathbf{v}$ is the vector of the measurement noise. Here we assume that all noise is an uncorrelated Gaussian distribution with zero mean and fixed covariance as described by $\mathbf{v}_x\sim\mathbf{N}(\mathbf{0},\mathbf{R}_x)$ and $\mathbf{v}_y\sim\mathbf{N}(\mathbf{0},\mathbf{R}_y)$. Hence let us design an extended Kalman filter (EKF) [23] to substitute the deterministic observers in Eqs. (10b) and (11b) and to reject the noise as shown in Fig. 4. Evidently, the design procedures of the synchronization are similar to Eqs. (10a), (10b), (11a), and (11b), but different in the observer gain $(\mathbf{K}_x,\mathbf{K}_y)$ selection. Accordingly, the error dynamics of the synchronization is the same as in Eq. (13), while the estimated error dynamics of the EKF are slightly different from Eq. (12) as

$$\begin{aligned}\dot{\mathbf{e}}_x&=(\mathbf{A}-\mathbf{K}_x\mathbf{C})\mathbf{e}_x+f(\mathbf{x}+\mathbf{e}_x)-f(\mathbf{x})-\mathbf{K}_x\mathbf{v}_x,\\ \dot{\mathbf{e}}_y&=(\mathbf{A}-\mathbf{K}_y\mathbf{C})\mathbf{e}_y+f(\mathbf{y}+\mathbf{e}_y)-f(\mathbf{y})-\mathbf{K}_y\mathbf{v}_y.\end{aligned} \tag{15}$$

Surely, in Eq. (15), based on the criterion of the EKF, a minimum variance result is obtained. According to linearization about the current estimated states, the EKF's gain must be computed in real time as

$$\dot{\mathbf{P}}_x=(\mathbf{A}+\mathbf{D}f_x)\mathbf{P}_x+\mathbf{P}_x(\mathbf{A}+\mathbf{D}f_x)^T-\mathbf{P}_x\mathbf{C}^T\mathbf{R}_x^{-1}\mathbf{CP}_x, \tag{16a}$$

$$\mathbf{K}_x=\mathbf{P}_x\mathbf{C}^T\mathbf{R}_x^{-1}, \tag{16b}$$

$$\dot{\mathbf{P}}_y=(\mathbf{A}+\mathbf{D}f_y)\mathbf{P}_y+\mathbf{P}_y(\mathbf{A}+\mathbf{D}f_y)^T-\mathbf{P}_y\mathbf{C}^T\mathbf{R}_x^{-1}\mathbf{CP}_y, \tag{17a}$$

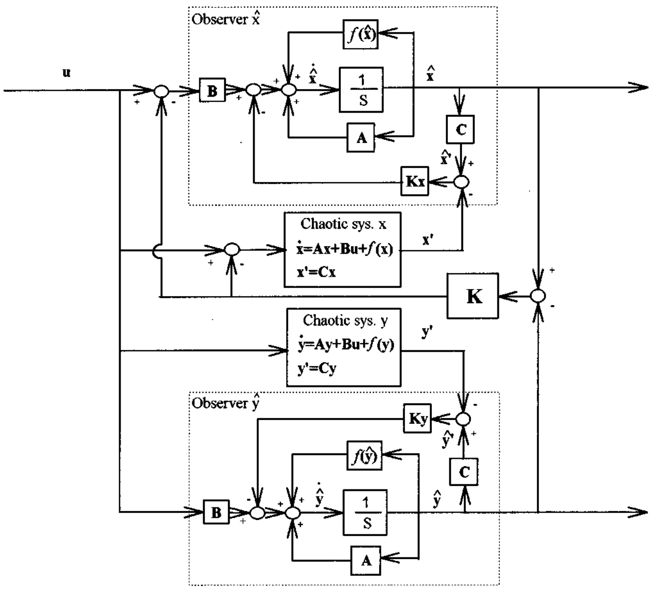

FIG. 3. Synchronization of two chaotic systems aids by two deterministic observers.

$$\mathbf{K}_y=\mathbf{P}_y\mathbf{C}^T\mathbf{R}_y^{-1}, \tag{17b}$$

where

$$\mathbf{D}f_x=\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{x=\hat{x}}, \quad \mathbf{D}f_y=\left.\frac{\partial f(\mathbf{y})}{\partial \mathbf{y}}\right|_{y=\hat{y}}. \tag{18}$$

Clearly, Eq. (18) is a first-order approximation. A high-order expansion can be easily derived, but with more time consumption. Here the matrices $\mathbf{P}_x,\mathbf{P}_y$ are an approximation of the true covariance matrix and $\mathbf{K}_x,\mathbf{K}_y$ are calculated based on the rule of minimum variance. An EKF in engineering fields is not always stable and we have to allow for a small error in the initial estimation. Once the filters converge, the design procedures of synchronization are the same as the full state measurement procedure shown in Eq. (3). To this end, the synchronization with the observer can be summarized as follows.

Step 1. In order to cover any initial errors, a changeable maximum eigenvalue of $\mathbf{P}$ is needed (i.e., the eigenvalues of $\mathbf{A}_0$ can be assigned), which also means that the pairs $(\mathbf{A},\mathbf{B})$ and $(\mathbf{A},\mathbf{C})$ are controllable and observable, respectively (the requirement of a full measurement is substituted by an observable constrain). If this is not the case, we are aware that the initial error may not be given arbitrarily and the synchronization is stabilized case by case.

Step 2. From Eq. (5) the local Lipschitz constant L is evaluated according to the nonlinear function and the initial error within two systems.

Step 3. If the systems are not full state measurements and the pair $(\mathbf{A},\mathbf{C})$ is observable, then design the observers (deterministic or stochastic) for the systems with feedback gain $\mathbf{K}_x$ and $\mathbf{K}_y$.

Step 4. Selected the synchronization coupled gain $\mathbf{K}$ and then check whether or not the inequality (9) is satisfactory.

III. EXAMPLES

A. Example 1

The synchronization of two Lorenz systems [15] is studied in this example. On the basis of Eq. (2), the systems in the Lure form are described as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}-p & p & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}0\\ -x_1x_3\\ x_1x_2\end{pmatrix},$$
$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix}=\begin{pmatrix}-p & p & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}+\begin{pmatrix}0\\ -y_1y_3\\ y_1y_2\end{pmatrix}. \tag{19}$$

To synchronize, the systems are coupled in the form of Eq. (3) and are given by

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}-p & p & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}0\\ -x_1x_3\\ x_1x_2\end{pmatrix} -\mathbf{BKC}(\mathbf{x}-\mathbf{y}),$$
$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix}=\begin{pmatrix}-p & p & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}+\begin{pmatrix}0\\ -y_1y_3\\ y_1y_2\end{pmatrix}. \tag{20}$$

The parameters are chosen as $p=10$, $r=28$, and $b=8/3$; thereupon both systems are chaotic.

Step 1. To simplify the problem and coupled gain selection, in this first example let all the states be measured and three control inputs be considered, i.e., two identity matrices $\mathbf{B}=\mathbf{C}\in\mathbf{R}^{3\times3}$. Therefore, the coupled gain $\mathbf{K}$ will be a 3×3 matrix and the eigenvalues of $(\mathbf{A}-\mathbf{BKC})$ can be assigned arbitrarily. Note that, in the case of the Lorenz system, the three control inputs can easily be fulfilled by a circuit (see, e.g., [3,24]) and with two control inputs, the pair $(\mathbf{A},\mathbf{B})$ is controllable.

Step 2. Because $f(\mathbf{y})=(0,-y_1y_3,y_1y_2)^T$, the local Lipschitz constant L is evaluated by

$$\left\|\frac{\partial f(\mathbf{y})}{\mathbf{y}}\right\|_\infty=\begin{Vmatrix}0 & 0 & 0\\ -y_3 & 0 & -y_1\\ y_2 & y_1 & 0\end{Vmatrix}_\infty =\max(0,|y_3|+|y_1|,|y_2|+|y_1|)=L. \tag{21}$$

Let the initial error of the Lorenz systems be arbitrary. Roughly, in the state space we have

$$|y_1|\leqslant 60, \quad |y_2|\leqslant 60, \quad |y_3|\leqslant 60. \tag{22}$$

Substituting Eq. (22) into Eq. (21) and taking the worse case of the initial error, i.e., $\tilde{y}_1=\tilde{y}_2=\tilde{y}_3=120$, we have $L=240$.

Step 3. There is no need for an observer.

Step 4. Select the coupled gain to be $\mathbf{K}=(300,0,0;\ 0,300,0;\ 0,0,300)^T$, with an identity matrix $\mathbf{C}$, and the eigenvalues of $\mathbf{A}_0$ -322.8, -288.2, and -302.7, respectively. From the Lyapunov equation (6) we obtain

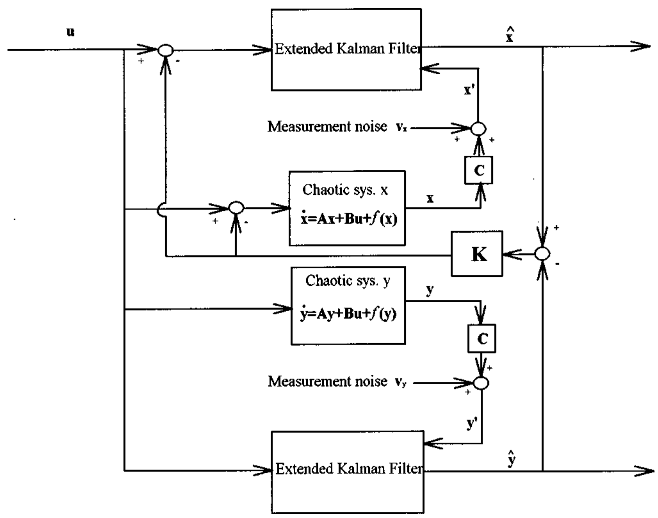

FIG. 4. Synchronization of two noisy chaotic systems aids by two stochastic observers (EKF).

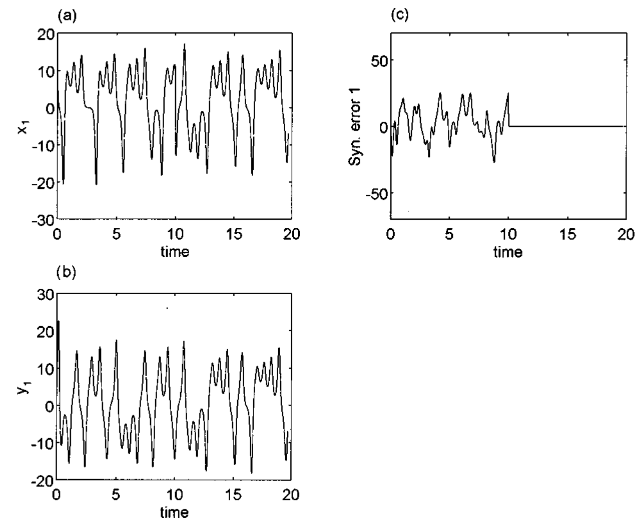

FIG. 5. Synchronization of the Lorenz systems with the command switched on at $t=10$ sec: (a) state x_1, (b) state y_1, and (c) synchronization error x_1-y_1.

$$\mathbf{P}=\begin{pmatrix} 0.0016 & 0.0001 & 0 \\ 0.0001 & 0.0017 & 0 \\ 0 & 0 & 0.0017 \end{pmatrix}, \quad \lambda_{\max}(\mathbf{P})=0.0015.$$

Thus the inequality (9) is satisfactory because $2\lambda_{\max}(\mathbf{P})L=2\times 0.0015\times 240=0.72<1$. The results of the simulation are shown in Fig. 5 and the synchronization command was switched on at $t=10$ sec. Owing to a high coupled gain feedback, it can be seen that the synchronization error converged instantly.

B. Example 2

The Duffing system [13] in the form of Eq. (2) is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix} b\ \cos(t)+\begin{pmatrix} 0 \\ -x_1^3 \end{pmatrix} -\mathbf{BKC}(\mathbf{x}-\mathbf{y}), \tag{23}$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix} b\ \cos(t)+\begin{pmatrix} 0 \\ -y_1^3 \end{pmatrix},$$

where the parameters are chosen as $a=0.1$ and $b=10$. Thus these systems are chaotic. It can be easily verify that the pair $(\mathbf{A},\mathbf{B})$ is controllable. In order to compare with the existing literature [13,14], we first design the synchronization problem without the observer and then with the observer.

1. The output feedback coupling

Step 1. We use one control input (physical constraint) and a single measurement (position) in this example, i.e., $\mathbf{B}=(0,1)^T$ and $\mathbf{C}=(1,0)$, where the pair $(\mathbf{A},\mathbf{C})$ is observable. Accordingly, this is neither a full state measurement nor an observer assistance case and the initial error in Eq. (13) cannot be given arbitrarily.

Step 2. Because $f(\mathbf{y})=(0,-y_1^3)^T$, the local Lipschitz constant L is evaluated by

$$\left\| \frac{\partial f(\mathbf{y})}{\mathbf{y}} \right\|_\infty=\begin{pmatrix} 0 & 0 \\ -3y_1^2 & 0 \end{pmatrix}_\infty=\max(0,|3y_1^2|)=L. \tag{24}$$

If the initial errors of the two synchronization systems are confined by $|y_1|\leqslant 0.1$ and $|y_2|\in R$, because y_1 is a measured state, it is still possible to achieve this initial error. Therefore, $L=0.03$.

Step 3. This step is passed over.

Step 4. Select the coupled gain to be $K=1$. Then the eigenvalues of $\mathbf{A}_0$ are $-0.05\pm 0.998i$. From the Lyapunov equation (6), we have

$$\mathbf{P}=\begin{pmatrix} 10.05 & -0.5 \\ -0.5 & 10 \end{pmatrix}, \quad \lambda_{\max}(\mathbf{P})=10.526.$$

Thus the inequality (9) is satisfactory because $2\lambda_{\max}(\mathbf{P})L=2\times 10.526\times 0.03=0.63<1$.

We omitted the numerical simulation of this case because a long synchronization time is needed and the results can be seen in [13,14]. Nevertheless, because of the low-dimensional Duffing system, we can easily look closely at the convergent factors mentioned above. In this example, based on Eq. (4), the error dynamics of the synchronization is

$$\begin{aligned}\dot{\tilde{\mathbf{y}}}&=(\mathbf{A}-\mathbf{BKC})\tilde{\mathbf{y}}+f(\mathbf{y}+\tilde{\mathbf{y}})-f(\mathbf{y})\\ &=\left[\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}-\begin{pmatrix} 0 \\ 1 \end{pmatrix}K(1,0)\right]\tilde{\mathbf{y}}+f(\mathbf{y}+\tilde{\mathbf{y}})-f(\mathbf{y})\\ &=\begin{pmatrix} 0 & 1 \\ -K & -a \end{pmatrix}\tilde{\mathbf{y}}+f(\mathbf{y}+\tilde{\mathbf{y}})-f(\mathbf{y}).\end{aligned} \tag{25}$$

Clearly, the eigenvalues of $\mathbf{A}_0$ are $\lambda_{1,2}=(-a\pm\sqrt{a^2-4K})/2$. The coupled gain K cannot arbitrarily change the eigenvalues of $\mathbf{A}_0$, but does influence the matrix $\mathbf{P}$ a little. Notably, in Eq. (25) the synchronization error under the nonlinear term perturbation $f(\mathbf{y}+\tilde{\mathbf{y}})-f(\mathbf{y})$ converges on its own. Clearly, a larger initial error will eventually diverge the error equation (25).

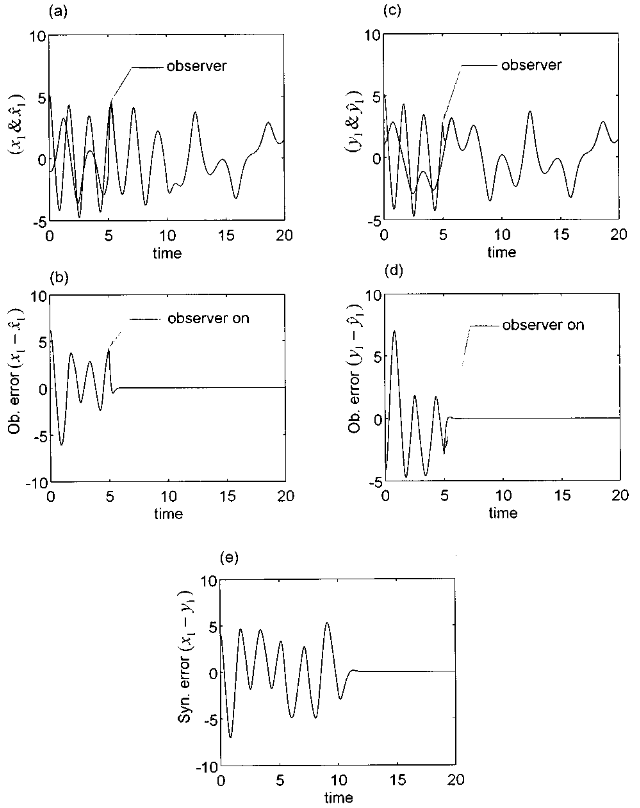

FIG. 6. Synchronization of the Duffing systems (deterministic case) with the observer switched on at $t=5$ sec and synchronization command switched on at $t=10$ sec: (a) state x_1 and the observer's state $\hat{x}_1$, (b) state y_1 and the observer's state $\hat{y}_1$, and (c) synchronization error x_1-y_1.

2. The observer feedback coupling (deterministic case)

Now let us reconsider those steps in the above discussion with a deterministic observer.

Step 1. The pairs (**A**,**B**) and (**A**,**C**) are controllable and observable, respectively.

Step 2. Let the initial error of the Duffing systems be arbitrary. Roughly, in the state space we have

$$|y_1|\leqslant 5, \quad |y_2|\leqslant 20. \tag{26}$$

Substituting Eq. (26) into Eq. (24) and taking the worse case of the initial error, i.e., $\tilde{y}_1=10$ and $\tilde{y}_2=40$, we have $L=75$.

Step 3. Set $\mathbf{K}_x=\mathbf{K}_y=[100,1000]^T$; based on the inequality (9) the observers are converged. Therefore, we have $\lambda_{\max}(\mathbf{P})=0.005$.

Step 4. Select the coupled gain to be $\mathbf{K}=(100,50)$. Inequality (9) holds. The numerical simulations are shown in Fig. 6 with the observer switched on at $t=5$ sec and synchronization command switched on at $t=10$.

3. The observer feedback coupling (stochastic case)

The synchronization problems in Eq. (23) with the measurement noises are rewritten as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & -a\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}b\ \cos(t)+\begin{pmatrix}0\\ -x_1^3\end{pmatrix}$$
$$-\mathbf{BK}(\mathbf{x}'-\mathbf{y}'), \quad \mathbf{x}'=\mathbf{Cx}+\mathbf{v}_x \tag{27}$$

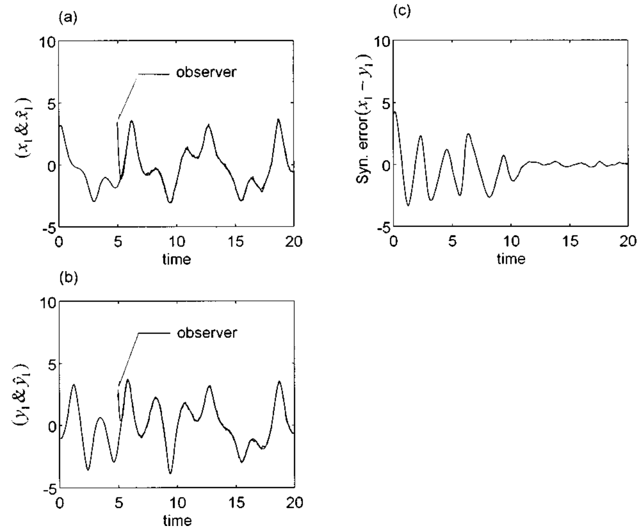

FIG. 7. Synchronization of the Duffing systems under noisy measurements. Deterministic observers are applied: (a) state x_1 and the estimated state $\hat{x}_1$, (b) state y_1 and the estimated state $\hat{y}_1$, and (c) synchronization error x_1-y_1.

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & -a\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}b\ \cos(t)+\begin{pmatrix}0\\ -y_1^3\end{pmatrix},$$
$$\mathbf{y}'=\mathbf{Cy}+\mathbf{v}_y.$$

In this noisy measurement case, we simply assume that the noise is uncorrelated Gaussian data (white noise) with mean and covariance $\mathbf{v}_x\sim\mathbf{N}(\mathbf{0},\mathbf{R}_x)$ and $\mathbf{v}_y\sim\mathbf{N}(\mathbf{0},\mathbf{R}_y)$, where

$$\mathbf{R}_x=\begin{pmatrix}0.2^2 & 0\\ 0 & 0\end{pmatrix}, \quad \mathbf{R}_y=\begin{pmatrix}0.2^2 & 0\\ 0 & 0\end{pmatrix}.$$

Hence the design procedures are exactly the same as those in the deterministic case except that the feedback gains $(\mathbf{K}_x,\mathbf{K}_y)$ are time-varying data and have to be calculated at

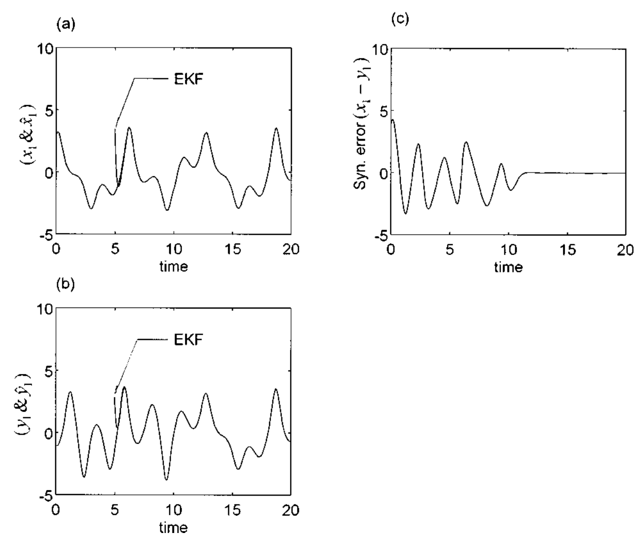

FIG. 8. Synchronization of the Duffing systems under noisy measurements. Extended Kalman filters are applied: (a) state x_1 and the estimated state $\hat{x}_1$, (b) state y_1 and the estimated state $\hat{y}_1$, and (c) synchronization error x_1-y_1.

each time step according to Eqs. (17) and (18). The simulation results in this case are shown in Figs. 7 and 8. First, in Fig. 7 we show the results of synchronization with only the aid of deterministic observers. As shown in Fig. 7(c), there exists a steady-state error in synchronization, which depends on the intensity of the external noises. Thereafter, in Fig. 8 an EKF was applied. In Fig. 8(c) the noise has been effectively rejected and the error of synchronization is amply ameliorated. In addition, in the EKF design, the noise is not required to be uncorrelated or certain; the more the information implies, the smaller the estimation errors and the faster the EKF convergence.

IV. CONCLUSION

In summary, we have presented a systematic procedure to analyze the state feedback control method for achieving synchronization. In this study the problems of synchronization (i.e., gain selection, the number of measurements, and the initial error within two chaotic systems) have been clarified. By a methodical analysis, our results point out that if the initial error is arbitrary, then, theoretically, a full state measurement is needed; otherwise there is an inequality that must hold. Furthermore, with the aid of the observers, synchronization that is only output coupled (not a full state measurement) or under a noisy measurement still can be efficiently accomplished.

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